Motivation

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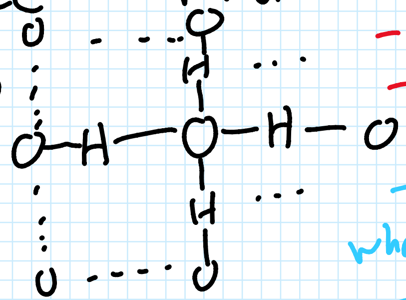
Water = 2 H, 1 O (tech should be 3D hexagonal)

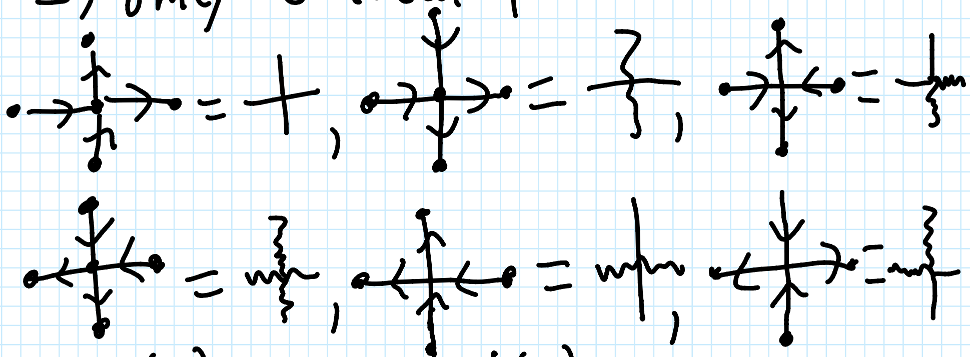
Ice = Water arranged in a crystal (lattice)

⇒ 

- each bond has a H
- only 2 H near 1 O
- Let > and < denote where H is on edge

— • = O

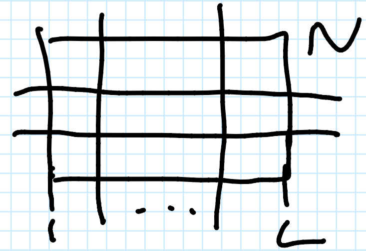
⇒ only 6 local pictures



a(u) b(u) c(u)

↓ or ← = ∿∿, ↑ or → = —

Def The 6 vertex model is the 2D stat mech system where sites = edges of a $L\times N$ lattice



σ_e = ∿∿∿ or —
$|\leftarrow\rangle$ $|\rightarrow\rangle$
$\begin{pmatrix}1\\0\end{pmatrix}$ $\begin{pmatrix}0\\1\end{pmatrix}$

w/ partition function

# of b in σ

$$Z_{L\times N}(u) = \sum_{\{\sigma\}} a(u)^{\sigma(a)}\, b(u)^{\sigma(b)}\, c(u)^{\sigma(c)}$$

Remark:	6 vertex	2D Ising
sites:	- on edges	- on vertices
Hamil contr:	- from vertices	- from edges

Remark: $a=b=c$ (square ice), $a>b=c$ (KH_2SO_4)

Goal: Find $Z_{L\times N}(u)$ w/ periodic boundary conditions

6 vertex

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Consider $R(u) = \begin{pmatrix} a(u) & & & \\ & b(u) & c(u) & \\ & c(u) & b(u) & \\ & & & a(u) \end{pmatrix}$

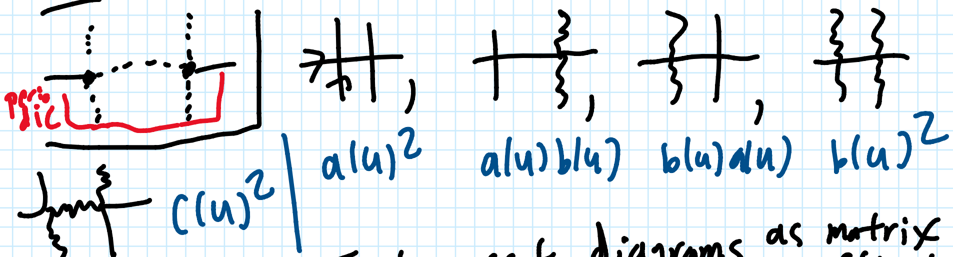
$R(u)|--\rangle = a(u)|--\rangle$

$R(u)|++\rangle = a(u)|++\rangle$

$R(u)|-+\rangle = b(u)|-+\rangle + c(u)|+-\rangle$

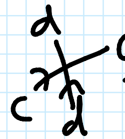
$R(u)|+-\rangle = c(u)|-+\rangle + b(u)|+-\rangle$ — undef

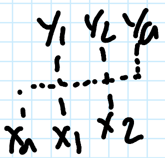
Ex: $L=2, N=1$. Find all states s.t.

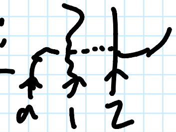


Interpret diagrams as matrix coefficients

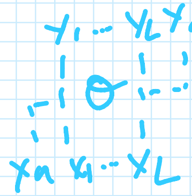
Key Insight:

- Label  $= R(u)_{cd}$

 $= \langle y_a y_1 y_2 | T | x_a x_1 x_2 \rangle$

Ex:  $= \langle +-+ | R(u)_{a2} R(u)_{a1} | +-+ \rangle$

Lemma

$\sum wt$  $= \langle y_a y_1 \cdots y_L | R(u)_{aL} \cdots R(u)_{a1} | x_a x_1 \cdots x_L \rangle$

top $\vec{\theta} = y_1 \cdots y_a$

bot $\vec{\theta} = x_a \cdots x_L$

Pf by Ex: $R(u)_{a2} R(u)_{a1} |+-+\rangle$

$= R(u)_{a2}(c(u)|-++\rangle + b(u)|+-+\rangle)$

$= c(u)\, b(u)|-++\rangle + c(u)^2|++-\rangle$

$+ b(u)\, a(u)|+-+\rangle$

$\langle +-+ | R(u)_{a2} R(u)_{a1} | +-+\rangle = b(u)a(u)$ ✓

$\Rightarrow \langle ++- | R(u)_{a2} R(u)_{a1} | +-+\rangle = c(u)^2$ ✓

$\Rightarrow \langle ++- | R(u)_{a2} R(u)_{a1} | +-+\rangle = c(u)^2$

B/c periodic endpoints $\Rightarrow x_a = y_a$

$\Rightarrow$ wts are encoded in $\mathrm{Tr}_a(R_{aL}(u) \ldots R_{a1}(u))$ as matrix coeff

$\tilde{m}(u) = \mathrm{Tr}_a(\ldots)$, $= t(u)$

Commutativity

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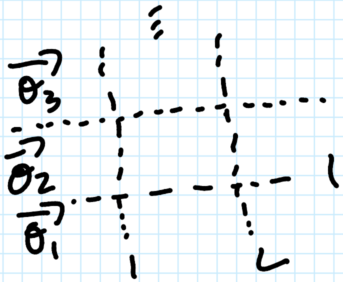


$\Rightarrow Z_{L\times N}(u)$

$= \sum_{\{\vec\theta\}} \langle \vec\theta_1 | t(u) | \vec\theta_N\rangle \cdots \langle\vec\theta_3|t(u)|\vec\theta_2\rangle\langle\vec\theta_2|t(u)|\vec\theta_1\rangle$

$= \mathrm{Tr}(t(u)^N)$ so need to find e.v. for $t(u)$

Recall $M_a(u) = L_{aL}(u)\cdots L_{a1}(u)$ in ABA and $T(u) = \mathrm{Tr}_a M_a(u)$.

$$M_a(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

Recall $B(u_1)\cdots B(u_k)|\uparrow^L\rangle$, $k \le L$ are e$\vec v$ for $T(u)$ w/ e.v $\tau(u|u_k)$

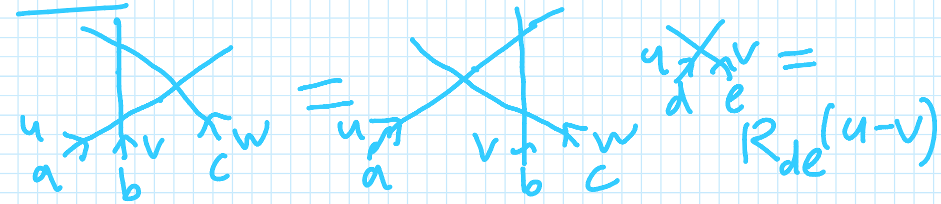
Remark: It turns out $R_{ak}(u) = L_{ak}(u+\frac{i}{2})$

$\Rightarrow t(u) = T(u+i/2) \to$ know e.v for $t(u)$

Claim: H_{xxx}, $t(u)$ have the same e.$\vec v$.

Pf: True if $H_{xxx}t(u) = t(u)H_{xxx}$

Lemma 1: $R(u)$ satisfies the Yang-Baxter eq



$R_{ab}(u-v)R_{ac}(u-w)R_{bc}(v-w)$

$= R_{bc}(v-w)R_{ac}(u-w)R_{ab}(u-v)$

Remark: Set $w=0$, $u \mapsto u-i/2$, $v \mapsto v-i/2$

YB $\Rightarrow$ RLL relation

Prop 2: $R_{ab}(u-v)\tilde{m}_a(u)\tilde{m}_b(v) = \tilde{m}_b(v)\tilde{m}_a(u)R_{ab}(u-v)$

Lemma 3:
$$\begin{tikzcd} \mathrm{Mat}(\mathbb{C}^2_a \otimes \mathbb{C}^2_b \otimes H) \arrow[rr, "\mathrm{Tr}_{a,b}"] \arrow[dr, "\mathrm{Tr}_a"'] & & \mathrm{Mat}(H) \\ & \mathrm{Mat}(\mathbb{C}^2_a \otimes H) \arrow[ur, "\mathrm{Tr}_b"'] & \end{tikzcd}$$

$\mathrm{Tr}_{a,b} = \mathrm{Tr}_a \circ \mathrm{Tr}_b$

Commutativity II

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Lemma 4: Let $M, N \in Mat(\mathbb{C}^2_a \otimes H)$ and suppose N doesn't act on $\mathbb{C}^2_a$. Then

$$Tr_a(MN) = Tr_a(M) \circ N$$

Pf: N not acting on $\mathbb{C}^2_a \Rightarrow$ "N" $= I_2 \otimes N$

Let $M = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ $A_{ij} \in Mat(H)$ $\begin{pmatrix} N & 0 \\ 0 & N \end{pmatrix}$

$$\Rightarrow Tr_a(MN) = Tr_a\left(\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} N & 0 \\ 0 & N \end{pmatrix}\right)$$

$$= Tr_a\begin{pmatrix} A_{11}N & A_{12}N \\ A_{21}N & A_{22}N \end{pmatrix} = A_{11}N + A_{22}N$$

$$= Tr_a(M)N$$

Theorem 5: If $R_{ab}(u-v)$ is invertible

$$t(u)t(v) = t(v)t(u) \Rightarrow \text{ $t(v)t(u)$ have same e.}\vec{v}$$

Pf: $Tr_a(Tr_b(M_a(u)M_b(v))) \overset{L3}{=} Tr_{a,b}(M_a(u)M_b(v))$

$\overset{L4}{"}\ t(u)t(v)$

$\overset{RMM}{=} Tr_{a,b}(R_{ab}(u-v)^{-1}M_b(v)M_a(u)R_{ab}(u-v))$

$\overset{cyc}{=} Tr_{a,b}(M_b(v)M_a(u)R_{ab}(u-v)R_{ab}(u-v)^{-1})$

$= Tr_{a,b}(M_b(v)M_a(u)) \overset{L3}{=} Tr_a(Tr_b(M_b(v)M_a(u)))$

$\overset{L4}{=} Tr_a(Tr_b(M_b(v))M_a(u)) \overset{L4}{=} Tr_b(M_b(v))Tr_a M_a(u)$

$= t(v)t(u).$

Lemma 6: Let $T(u)\vec{v} = \lambda(u)\vec{v}$

(a) $T(u)^{-1}\vec{v} = \lambda(u)^{-1}\vec{v}$

(b) $T'(u)\big|_{u=u_0}\vec{v} = \lambda'(u)\big|_{u=u_0}\vec{v}$

$\Rightarrow$ $t(u)^{-1}, t'(u)|_{u=u_0}$ commute w/ $t(v)$ "$\forall$" u, v

Pf: (b) $\frac{d}{du}$ is linear

Thrm 7: $H_{xxz}t(v) = t(v)H_{xxz}$

Pf: Recall H_{xxz} "=" $\frac{d}{du}\log t(u)\big|_{u=0}$

$H_{xxz}t(v) = t'(0)t(0)^{-1}t(v) \overset{L6}{=} t'(0)t(v)t(0)^{-1}$

$\overset{L6}{=} t(v)t'(0)t(0)^{-1} = t(v)H_{xxx}$

R-matrix

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- Thus CBA/ABA solves 2 problems

(1) Partition function of 6 vertex (2D stat)

(2) Schrodinger eq of H_{XXZ} (1D quantum)

Overview of ABA

YB eq $\Rightarrow$ RLL $\Rightarrow$ RMM

$\Rightarrow$ Relations in "Yang-Baxter alg"

- $B(u)B(v)=B(v)B(u)$ etc

H_{XXZ} and $t(u)$ have same e.v $\vec{v}$

$\Rightarrow B(u_1)\dots B(u_k)|\uparrow\rangle$ are the e.v for $t(u)$

- Suppose $R(u)\in Mat(V\otimes V)$ satisfies the YB eq and $\exists u_0\in\mathbb{C}$ s.t. $R(u_0)=P$, $P(\vec{v}\otimes\vec{w})=\vec{w}\otimes\vec{v}$.
- Then consider the quantum spin chain

$V\otimes\cdots\otimes V$

and let $\hat{H}=\sum \hat{H}_{n,n+1}$

$$\hat{H}_{n,n+1}=R'_{n,n+1}(u_0)R_{n,n+1}(u_0)^{-1}$$

Then $e\vec{v}$, ev of $\hat{H}$ can be solved via ABA

Def A quantum mech system $(V,\hat{H})$ is said to be integrable if $\exists R(u)$ satisfying the YB eq s.t. $\hat{H}$ = analytic function in $R(u)$

Q: where to find R-matrices?

A: Quantum groups

WARNING: Quantum groups are not groups!

Def A lie alg $(\mathfrak{g},[\,,\,])$ is a v.s. $\mathfrak{g}$ w/ $[\,,\,]:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ satisfying .....

Ex: $\mathfrak{g}=M_{n\times n}(\mathbb{C})$, $[A,B]=AB-BA$

Def A quantum group $U_q(\mathfrak{g})$ is a q-deformation of a lie alg $\mathfrak{g}$ ($U_q(\mathfrak{g})\xrightarrow{q\to 1}\mathfrak{g}$)

Fact: quantum groups come equipped w/ "universal R-matrices" $R_{un}\in U_q(\mathfrak{g})\otimes U_q(\mathfrak{g})$ s.t.

R_{un} satisfies the YB eq

Def A representation (ρ, V) of $U_q(\mathfrak{g})$ is a ring homomorphism

$$\rho : U_q(\mathfrak{g}) \longrightarrow Mat(V)$$

- Then for any representation (ρ, V) of $U_q(\mathfrak{g})$

$$\rho(R_{un}) \in Mat(V) \otimes Mat(V) \cong Mat(V \otimes V)$$

will satisfy the YB eq

$\rightarrow$ this gives us a lot of R-matrices!

Upshot: Different reps of $U_q(\mathfrak{g})$ give rise to different models that are solved w/ same algebra.

Ex: XXX and Lieb-Liniger are different rep of the Yangian $Y(Mat_{2\times 2}(\mathbb{C}))$

Review: ☆ 1D Ising ① Solved for Z_N, $\langle \sigma_r \sigma_t \rangle$ via recursive (no YB), transfer matrix $Z_N = Tr(V^N)$

② Potts model + series expansion of Z_N

③ $O(n)$ model + Bessel functions

④ Z_N-model + Gamma Function

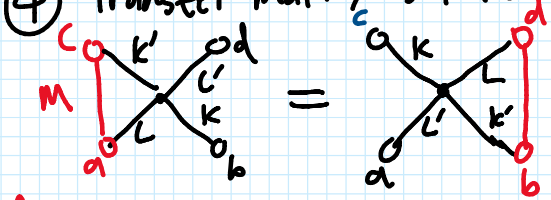
☆ 2D Ising ① duality in high + low temp series exp $\Rightarrow$ has a phase transition

② Pfaffians, $Pf(A^{o}_{G}) = \#$ dimers on G

$Pf(A) = \sqrt{\det A}$, $Z_N = \#$ dimers on Fischer lattice

③ Circulant matrices, e.v, $e\vec{v}$ + explicit formula for # dimers on $m \times n$ lattice

④ Transfer matrix sol to 2D Ising



"star-triangle eq"
local equality of Z_N
(Secretly Yang-Baxter)

☆ Quantum Integrability ① Tensor products, Quantum computing, Heisenberg spin chains

② CBA sol to XXX spin chain

Review

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(3) ABA sol to XXX spin chain

(4) 6 vertex $\underbrace{\text{model, YB and}}_{\text{Quantum Groups}}$ R-matrices